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ON THE EXTREMAL FUNDAMENTAL FREQUENCIES OF VIBRATING BEAMS†

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Abstract—Bernoulli–Euler Beams with variable cross section are optimized with respect to their fundamental frequency of transverse oscillation. The cross section is allowed to vary in a manner such that the second area moment is linearly related to the area. Using calculus of variations, the fundamental frequency is made stationary. The closed-form solution is found for all sets of homogeneous boundary conditions. In most cases, the resulting beam is uniform, however, the frequency in some cases is a minimum, others a maximum. For cantilever and free-free beams no maximum fundamental frequency exists for this type of cross section variation.

1. INTRODUCTION

THERE has been much effort recently in the area of the optimum design of vibrating elastic elements. Niordson [1] has shown that for simply supported beams with similar cross sections, the fundamental frequency of transverse vibration can be maximized by tapering the beam. The maximum value is about 6·6 % larger than the corresponding uniform beam. In [2], Brach considers the minimum transient response of beams. Starting with the Action Integral, Taylor [3] has developed the Euler equations for a sandwich cantilever beam whose solution furnishes the stationary value of the fundamental frequency. He gives the solution when the structural mass is very small compared to a distribution dead (non-structural) mass. Also in [3] as well as Turner [4], the maximum fundamental frequency of an elastic bar vibrating axially is considered.

In all of the previous work concerning transverse vibration, the second area moment is assumed to be proportional to either the first, second, or third power of the area (or mass). This excludes many practical sections such as I-beams and hollow sections. In this paper a linear relationship is used; this is discussed in Section 2. The extremal values of the fundamental frequency are found using variational calculus techniques. The closed-form solution of the resulting Euler equations is found for all types of homogeneous boundary conditions. The solution reveals that the extremal frequencies in some cases are minima, sometimes maxima and in some cases neither. Bernoulli–Euler beam theory is used.

2. CROSS SECTIONAL GEOMETRY

From classical vibration theory a linear elastic beam can vibrate harmonically in any one (or linear combination) of an infinite number of characteristic shapes, W_n. The characteristic equation is

$$\frac{d^2}{d\xi^2}\left[EI(\xi)\frac{d^2W_n}{d\xi^2}\right]-\omega_n^2\rho A(\xi)W_n = 0, \quad n = 1, 2, \ldots \tag{1}$$

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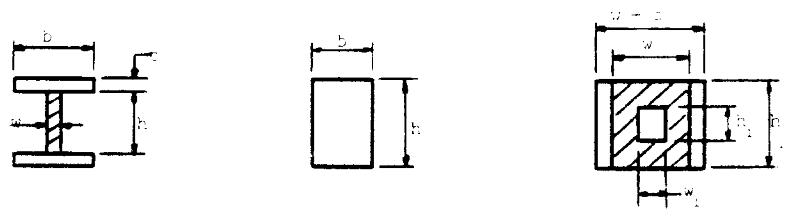
where

A = cross sectional area, E = modulus of elasticity, I = second area moment, L = length of the beam, W = transverse deflection, ξ = coordinate along the beam, $0 \le \xi \le L$, ρ = density of the beam material, and ω_n = eigenvalue.

It is not too difficult to show that if a rectangular cross section has a fixed width and variable height, the second area moment is proportional to the cube of the area. If the height and width vary with constant ratio, the second area moment is proportional to the square of the area. Finally if only the width varies, the proportion is linear. That is $I(\xi) = c_p A(\xi)^p$ where c_p is a constant implicitly depending upon p and $p = 1, 2$, or 3. If the cross section is not simply rectangular but does have a constant height the relationship is no longer proportional but does remain linear.

Let $x = \xi/L$ and $\alpha(x) = LA(x)/V$. In terms of these quantities, the linear area–area moment relationships are $\alpha(x) = \beta_0 + \beta(x)$ and $I(x) = c\gamma_0 + c\beta(x)$. The expression for α includes a fixed area, β_0, plus a variable portion, $\beta(x)$. The quantity $c\gamma_0$ is the second area moment corresponding to the fixed area β_0; $c\beta(x)$ is the second area moment corresponding to the variable area $\beta(x)$. Also, c is a constant. Some typical examples are shown in Table 1.

TABLE 1. EXAMPLE CROSS SECTIONS
area, $\alpha = \beta + \beta_0$; second area moment, $I = c\gamma_0 + c\beta$



$\beta = 2bt/V$	$\beta = bh/V$	$\beta = bh/V$
$\beta_0 = wh/V$	$\beta_0 = 0$	$\beta_0 = (wh - w_1h_1)/V$
$c = (4t^2 + 3h^2)V/12$	$c = Vh^2/12$	$c = vh^2/12$
$c\gamma_0 = wh^3/12$	$\gamma_0 = 0$	$c\gamma_0 = (wh^3 - w_1h_1^3)V/12$
$\gamma_0 < \beta_0$	$\gamma_0 = \beta_0$	$\gamma_0 > \beta_0$

Note: (a) For all cross sections shown the quantity b is permitted to vary along the span, other dimensions are constant; fixed area is cross hatched.
(b) The quantity V is the total volume of the beam; the length L is unity.

Both γ_0 and β_0 are positive since they are area moment and area respectively; the examples show that their ratio can be less than, equal to, or greater than one.

Using these relationships and the nondimensional quantities defined above, (1) can be written as

$$[(\beta + \gamma_0)W_n'']'' - \nu_n^2(\beta + \beta_0)W_n = 0, \quad n = 1, 2, \ldots \tag{2}$$

where $\nu_n^2 = \omega_n^2(\rho VL^3/Ec)$.

3. VARIATIONAL EQUATIONS

The problem to be solved here is to determine the extremal values of ν_1^2, the nondimensional fundamental natural frequency where $\nu_1^2 > 0$. If (2) is multiplied by W_1 and integrated over the range 0 to 1, an expression for ν_1^2 is obtained. This is

$$\nu_1^2 = \frac{\int_0^1[(\beta+\gamma_0)W_1'']''W_1\,dx}{\int_0^1(\beta+\beta_0)W_1^2\,dx} = \frac{\int_0^1(\beta+\gamma_0)(W_1'')^2\,dx}{\int_0^1(\beta+\beta_0)W_1^2\,dx}. \tag{3}$$

The second numerator is obtained if the first is integrated twice by parts and the following boundary conditions are used:

$$[(\beta+\gamma_0)W_1''][W_1'] = 0 \qquad \text{at } x = 0 \text{ and } 1$$

and (4)

$$[(\beta+\gamma_0)W_1'']'[W_1] = 0 \qquad \text{at } x = 0 \text{ and } 1.$$

All further results are for boundary conditions (4). In (3), ν_1^2 is a functional in the form of a quotient of two nonnegative integrals. The following method of deriving the necessary conditions for an extremal of ν_1^2 differs in technique from Niordson's method [1]. The following procedure is followed since it permits the introduction of multiple constraints in a more straight forward manner. It has been shown [5] that the stationary values of (3) correspond to the stationary values of a functional J, where

$$J = \int_0^1 [(\beta+\gamma_0)(W'')^2 - \nu^2(\beta+\beta_0)W^2]\,dx. \tag{5}$$

In (5) ν^2 is the stationary value of ν_1^2 and the subscript has been omitted from W_1.

In order to make the solutions (extremal values of the frequency) practical, it is necessary to add constraints to the problem. It is necessary to require that $\beta \geq 0$ since by definition, area must be positive. Actually, let $\beta \geq C_1 \geq 0$ where C_1 is an arbitrary constant. In addition, the amount of material in the beam is restricted to be finite: this leads to $\beta \leq C_2$; $C_2 > C_1$. In this case the constant C_2 can be evaluated by the integral

$$\int_0^1 (\beta+\beta_0)\,dx = 1. \tag{6}$$

This requires that the integral of the total area over the beam must equal the volume, V, where V is finite. These constraints will be enforced using the method of Lagrange multipliers. To transform them to equality constraints, two real variables $u(x)$ and $v(x)$ are defined such that

$$\beta - C_1 - u^2 = 0$$

and (7)

$$C_2 - \beta - v^2 = 0.$$

(Both constraints could be combined and treated with a single Lagrange multiplier [6] but the above approach yields slightly simpler expressions.)

At this point the problem is to find the stationary values of J subject to the constraints (7). The first variation of an augmented functional, J^*, with respect to each of the variables

β, W, u, and v is set to zero, to form the necessary conditions. The functional is

$$J^* = J + \nu^2 \int_0^1 [\lambda_u(\beta - C_1 - u^2) + \lambda_v(C_2 - \beta - v^2)]\, \mathrm{d}x.$$

where λ_u and λ_v are Lagrange multipliers. Since $\beta(x)$ and $W(x)$ are not independent (they are related through the characteristic equation, (2)), the variation with respect to these two variables must be simultaneous. This yields the first Euler equation:

$$\delta J^* = \int_0^1 \left[\frac{\partial F}{\partial \beta} + \frac{\partial F}{\partial W''}\delta W'' + \frac{\partial F}{\partial W}\delta W + (\lambda_u - \lambda_v)\delta\beta \right] \mathrm{d}x = 0 \tag{8}$$

where F is the integrand of (5). Note that

$$\int_0^1 \left[\frac{\partial F}{\partial W''}\delta W'' + \frac{\partial F}{\partial W}\delta W \right] \mathrm{d}x = \int_0^1 \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial W''}\right) + \frac{\partial F}{\partial W} \right] \delta W\, \mathrm{d}x.$$

This follows by twice integrating the first term by parts and applying the boundary conditions (4). It is required that the admissible functions satisfy the boundary conditions. Further note that

$$\int_0^1 \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial W''}\right) + \frac{\partial F}{\partial W} \right] \delta W\, \mathrm{d}x = 2\int_0^1 \{[(\beta + \gamma_0)W'']'' - \nu^2(\beta + \beta_0)\}\delta W\, \mathrm{d}x$$

By (2) this integrand is zero. This shows that the variation of J^* with respect to W to identically zero and thus can be disregarded. This was also found by Niordson [1] and Taylor [3]. As a result (8) reduces to

$$\delta J^* = \int \left[\frac{\partial F}{\partial \beta} + \nu^2(\lambda_u - \lambda_v) \right] \delta\beta\, \mathrm{d}x = 0.$$

Since this must be true for arbitrary variations, $\delta\beta$, of β, the integrand must be zero. This gives the first Euler equation:

$$(W'')^2 - \nu^2 W^2 + \nu^2(\lambda_u - \lambda_v) = 0 \tag{9}$$

This has the same form as equation (20) in [3]. The independent variations with respect to u and v furnish the remaining two Euler equations, respectively:

$$\lambda_u u = 0 \tag{10}$$

$$\lambda_v v = 0 \tag{11}$$

Assuming that stationary values exist, (9), (10), and (11) are the necessary conditions. They must be solved along with the characteristics equation (2), the boundary conditions (4) and the constraints (7).

4. SOLUTION

Before obtaining the solution, an expression can be found which furnishes some information concerning the Lagrange multipliers. Multiplication of (9) by $(\beta + \gamma_0)$ and integration (the first term again integrated by parts) leads to the relationship

$$\lambda_u - \lambda_v = (\gamma_0 - \beta_0)\frac{\int_0^1 W^2\,dx}{\int_0^1(\beta+\gamma_0)\,dx} \tag{12}$$

Thus when $\gamma_0 = \beta_0$ the first necessary condition, (9), reduces to

$$(W'')^2 - \nu^2 W^2 = (W'' + \nu W)(W'' - \nu W) = 0 \tag{9a}$$

This is used in the sequel.

The following possibilities can occur:

Case I: $\lambda_u \neq 0$ and $\lambda_v \neq 0$
Case II: $\lambda_u \neq 0$ and $\lambda_v = 0$
Case III: $\lambda_u = 0$ and $\lambda_v \neq 0$, or
Case IV: $\lambda_u = 0$ and $\lambda_v = 0$ or $\lambda_u - \lambda_v = 0$.

The first case yields an unrealistic solution since this requires that both constraints in (7) be zero simultaneously. Thus, in turn, means that $\beta = C_1$ and $\beta = C_2$ with $C_2 > C_1$. Clearly this cannot be a solution.

Case II. Since $\lambda_u \neq 0$, the necessary condition (10) implies that $u = 0$. From (7) it is seen that $\beta = C_1$. From (3), this indicates the extremal value of ν^2 is

$$\nu^2 = \frac{C_1+\gamma_0}{C_1+\beta_0}\frac{\int_0^1(W'')^2\,dx}{\int_0^1 W^2\,dx} \tag{3a}$$

Consider now case III.

Case III. Here $\lambda_v \neq 0$. From reasoning similar to the above, the extremal frequency is given by

$$\nu^2 = \frac{C_2+\gamma_0}{C_2+\beta_0}\frac{\int_0^1(W'')^2\,dx}{\int_0^1 W^2\,dx} \tag{3b}$$

Before discussing (3a) and (3b) consider the quotient

$$\frac{\int_0^1(W'')^2\,dx}{\int_0^1 W^2\,dx}.$$

For any given set of boundary conditions, this expression is a dimensionless number independent of the total area (and thus independent of C_1 or C_2)[7]. Therefore the extremal behavior depends only algebraically upon the quotient $f(C^*)$, where

$$f(C^*) = \frac{C^*+\gamma_0}{C^*+\beta_0}, \qquad C^* = C_1, C_2. \tag{13}$$

Figure 1 shows $f(C^*)$ for $\gamma_0 < \beta_0$. It is clear that (3a) is a minimum and (3b) is a maximum. On the other hand when $\gamma_0 > \beta_0$, Fig. 2 indicates that (3a) is a maximum while (3b) is a minimum. Recall that $C_1 \geq 0$; thus $C_1 = 0$ is an absolute minimum (or maximum depending on the relative magnitudes of γ_0 and β_0).

These results can be restated in the following way. The area β which yields a stationary frequency is a constant. When $\gamma_0 < \beta_0$, $\beta = C_2$ gives a maximum; $\beta = C_1$ gives a minimum. When $\gamma_0 > \beta_0$, $\beta = C_2$ gives a minimum, and $\beta = C_1$ gives a maximum. When $\gamma_0 = \beta_0$, the solution is treated separately as follows.

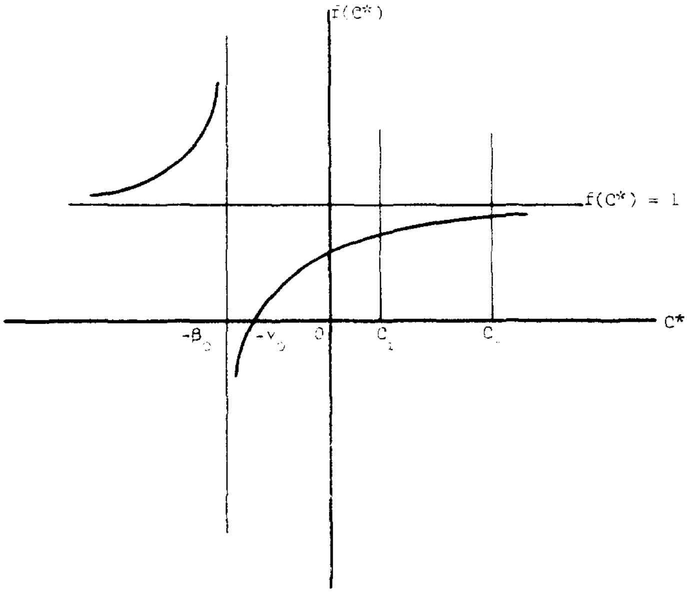


FIG. 1. Fundamental frequency, $\gamma_0 < \beta_0$.

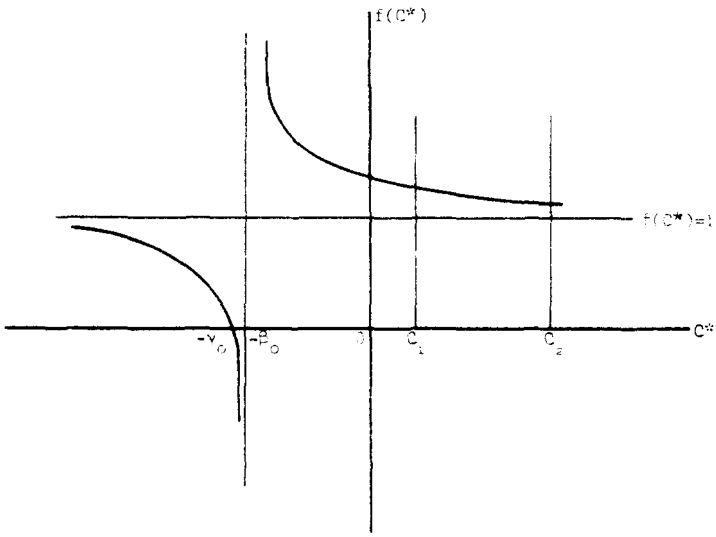


FIG. 2. Fundamental frequency, $\gamma_0 > \beta_0$.

Case IV. When $\gamma_0 = \beta_0$, $\lambda_u - \lambda_v = 0$ and the Euler equations reduce to (9a). This equation is the product of two linear equations,

$$W'' \pm \nu W = 0, \tag{14}$$

one equation for each sign. Using (14), W'' may be eliminated from the characteristic equation (2). This gives

$$[(\beta + \gamma_0)W]'' \pm \nu(\beta + \beta_0)W = 0.$$

If the derivative in the first term is expanded and (14) is used again, the result is

$$(\beta+\gamma_0)''W+2(\beta+\gamma_0)'W' = \beta''W+2\beta'W' = 0.$$

The second form occurs since γ_0 is a constant. The first integral of this equation is simply

$$\beta'W'^2 = a \tag{15}$$

where a is a constant. The value of a can be found using boundary conditions (4). Using (14) they can be put into the form

$$\pm\nu[(\beta+\gamma_0)W][W'] = 0 \quad \text{at } x = 0 \text{ and } 1$$

and (16)

$$\pm\nu[(\beta+\gamma_0)W]'[W] = 0 \quad \text{at } x = 0 \text{ and } 1.$$

The latter can be written $[\beta'W+(\beta+\gamma_0)W']W = 0$, where the factor $\pm\nu$ has been dropped. The two boundary conditions combined required that $\beta'W^2 = 0$ at $x = 0$ and $x = 1$. Thus $a = 0$. Since $W(x) = 0$ is considered a trivial solution. β' must be zero. This, in turn, shows that β = constant gives a stationary value to the frequency ν^2.

The value of β can be determined from (3). Since β is a constant, (3) shows that when the extremal exists, $\beta = 1-\beta_0$.

At this point a comment must be made concerning the boundary conditions arising from cantilever and free-free beams. The boundary conditions (16) arising from the use of (14) can be satisfied at the free ends only if the area is zero. Since β is constant, the solution is trivial. It is easy to demonstrate that there is no maximum frequency in these two cases. Consider a set of mass distributions, β, for a cantilever beam made up of a delta sequence, positive and non-increasing in $0 \leq x < 1$, and zero at $x = 1$. In the limit this sequence of shapes which satisfies the boundary condition approaches a delta function at the origin. From (3) the fundamental frequency approaches the ratio

$$[W''(0)]^2/[W(0)]^2.$$

This is unbounded and one can conclude that the frequency of a cantilever can be made as large as possible. It is also possible to construct a cantilever with a fundamental frequency lower than a uniform beam's frequency. Thus β = constant furnishes neither a maximum or a minimum in these cases. A similar demonstration can be made for a free-free beam.

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Абстракт—Оптимализируются балки Бернулли-Эйлера переменного сечения относительно их основной частоты поперечных колебаний. Поперечное сечение изменяется такимже путем, что второй момент сечения является линейной зависимостью поверхности сечения. Используя вариационное исчисление, определяется основная частота как стационарная. Находится решение, в замкнутом виде, для всей системы однородных граничных условий. В большинстве случаев результирующая балка оказывается однообразной, несмотря на то, что частота для некоторых случаев достигает минимум, а для других максимум. Оказывается, что консольных и свободных балок совсем не существует максимум основной частоты, при этом типе изменения поперечного сечения.